Logarithmic speeds for one-dimensional perturbed random walk in random environment

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Abstract

We study the random walk in random environment on $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, where the environment is subject to a vanishing (random) perturbation. The two particular cases that we consider are: (i) random walk in random environment perturbed from Sinai's regime; (ii) simple random walk with random perturbation. We give almost sure results on how far the random walker is from the origin, for almost every environment. We give both upper and lower almost sure bounds. These bounds are of order $(\log t)^\beta$, for $\beta \in (1, \infty)$, depending on the perturbation. In addition, in the ergodic cases, we give results on the rate of decay of the stationary distribution.

Key words and phrases: Random walk in perturbed random environment; logarithmic speeds; almost sure behaviour; slow transience.

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1 Introduction

The random walk in one-dimensional random environment in Sinai's regime (which we describe in detail below) is a famous example of a random walk with 'logarithmic speed': after a long time t, the random walk is, roughly speaking, about $(\log t)^2$ from the origin.

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In this paper we give other examples of random walks in random environments with logarithmic speeds; in these cases the environment is subject to a random perturbation.

Our results cover both recurrent and transient cases. In the models that we consider, the speed is, roughly speaking, of order $(\log t)^\beta$, where β depends upon the model. We shall see that for the models we consider, all $\beta \in (1, \infty)$ are attained. Examples of logarithmic transience for random walks (such as given in our Theorem 3 below) are seemingly rare. The terminology 'speed' is perhaps more natural in the transient case; in the recurrent case 'speed' can be thought of as a measure of the rate of growth of the upper envelope of the random walk. Before we give our main results, we describe the probabilistic setting in which we work.

Given an infinite sequence $\omega = (p_0, p_1, p_2, \ldots)$ such that, for some $\delta \in (0, 1/2)$, $\delta \leq p_i \leq 1 - \delta$ for all $i \in \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$, we consider $(\eta_t(\omega); t \in \mathbb{Z}^+)$ the nearest-neighbour random walk on $\mathbb{Z}^+$ defined as follows. Set $\eta_0(\omega) = r$ for some $r \in \mathbb{Z}^+$, and for $n \in \mathbb{N} := \{1, 2, \ldots\}$,

$$\begin{aligned} P[\eta_{t+1}(\omega) = n - 1 | \eta_t(\omega) = n] &= p_n, \\ P[\eta_{t+1}(\omega) = n + 1 | \eta_t(\omega) = n] &= 1 - p_n =: q_n, \end{aligned} \tag{1}$$

and $P[\eta_{t+1}(\omega) = 0 | \eta_t(\omega) = 0] = p_0$, $P[\eta_{t+1}(\omega) = 1 | \eta_t(\omega) = 0] = 1 - p_0 =: q_0$. The given form for the reflection at the origin ensures aperiodicity, which eases some technical complications.

We call the sequence of jump probabilities ω our *environment*. As an example, the case $p_i = 1/2$ for all i gives the symmetric simple random walk on $\mathbb{Z}^+$.

Here, we take ω itself to be random — then $\eta_t(\omega)$ is a *random walk in random environment* (RWRE). More precisely, $p_0, p_1, \ldots$ will be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We describe our particular model at the end of this section. The RWRE was first studied by Kozlov [17] and Solomon [28] (in the case where p_i, $i \geq 0$ form an i.i.d. sequence). There has been considerable interest in the RWRE recently; see for example [22] or [30] for surveys. Some authors (e.g. [27]) consider the RWRE with state space the whole of $\mathbb{Z}$. For our model we take the case of $\mathbb{Z}^+$, which gives rise to a richer set of models in the sense that we can obtain positive-recurrent behaviour.

An important case in which the random environment is homogeneous and in some sense critical is the so-called *Sinai's regime*. Here $(p_0, p_1, p_2, \ldots)$ is a sequence of i.i.d. random variables satisfying the condition $\mathbb{E}[\log(p_1/q_1)] = 0$, where $\mathbb{E}$ is expectation under $\mathbb{P}$. In this case, a result dating back to Solomon [28] says that $\eta_t(\omega)$ is null-recurrent for $\mathbb{P}$-almost every ω. Solomon's result shows that Sinai's regime is critical in the sense that, for an i.i.d. random environment, $\eta_t(\omega)$ is respectively ergodic (that is positive-recurrent, here) or transient as $\mathbb{E}[\log(p_1/q_1)] > 0$ or $\mathbb{E}[\log(p_1/q_1)] < 0$.

A notable property of the RWRE in Sinai's regime is its *speed* — roughly speaking $\eta_t(\omega)$ is of order $(\log t)^2$ for large t. One way to state this more precisely (for another, see the discussion in Section 2.3) is in terms of 'almost sure' behaviour, i.e. results that hold P-almost surely (a.s.) for $\mathbb{P}$-almost every (a.e.) ω. (For the remainder of this paper, we omit the P and $\mathbb{P}$ when the context is clear.) This is the kind of result we give in the present paper. In Sinai's regime for the RWRE on $\mathbb{Z}^+$, almost sure upper and lower bounds were given by Deheuvels and Révész ([4], Theorem 4 in particular). A similar upper bound result was given by Comets, Menshikov and Popov (see [1], Theorem 3.2), proved via a martingale technique related to some of the ideas in the present paper. Sharp results are given by Hu and Shi in [14]. In particular, the following iterated logarithm result follows from Theorem 1.3 of [14].

Theorem 1 *[14] Suppose that* $(p_0, p_1, p_2, \ldots)$ *is an i.i.d. sequence with* $\mathbb{E}[\log(p_1/q_1)] = 0$ *and* $\mathrm{Var}(p_1) > 0$*. Then there exists a constant* $K \in (0, \infty)$ *(given explicitly in [14]) such that, for a.e.* ω*, a.s., for any* $\varepsilon > 0$*,*

(i) for all but finitely many t

$$\frac{\eta_t(\omega)}{(\log t)^2} \leq (1 + \varepsilon) K \log \log \log t;$$

(ii) for infinitely many t

$$\frac{\eta_t(\omega)}{(\log t)^2} \geq (1 - \varepsilon) K \log \log \log t.$$

Note that 'a.e. ω' is short for '$\mathbb{P}$-almost every environment ω', and 'a.s.' is short for 'P-almost surely'. We use this shorthand in the statements of all our results. Our methods do not enable our results to be as sharp as those in [14]; the best that we obtain in Sinai's regime is included in Theorem 4 below. However, we obtain a much wider array of results.

We remark that a range of *polynomial* speeds can be attained in certain transient homogeneous random environment regimes (see e.g. [16]). In this paper we are interested primarily in *logarithmic* speed results like Theorem 1, for random environments that are *asymptotically* homogeneous. Our main results are almost sure upper bounds for $\eta_t(\omega)$ that are valid for a.e. ω and all but finitely many t, and almost sure lower bounds for $\eta_t(\omega)$ that are valid for a.e. ω *either* for all but finitely many t (if $\eta_t(\omega)$ is transient, see e.g. Theorem 3) *or* for infinitely many t (if $\eta_t(\omega)$ is recurrent, see e.g. Theorem 2). These bounds are all of size $(\log t)^\beta$, for some $\beta \in (1, \infty)$ that is a function of α (the size of the perturbation), depending on the model in question, with higher order logarithmic corrections.

We study two particular cases of random environment. In the first, our environment will be a perturbation of the i.i.d. environment of Sinai's regime (see Section 2.1). In the second, our environment will be a random perturbation of the simple symmetric random walk (see Section 2.2). By studying a range of perturbations, we obtain a spectrum of possible behaviour.

The related paper [21] employs the method of Lyapunov functions (see [6]) to give qualitative characteristics for these models (amongst somewhat more general results): specifically, criteria for recurrence, transience and positive-recurrence (ergodicity, here). In the present paper we are concerned with corresponding *quantitative* behaviour: specifically, speeds (for those cases with logarithmic speed) and, secondarily, the rates of decay of the stationary distribution in the ergodic cases identified in [21]. We summarize the relevant results from [21] at convenient points in Section 2 below.

The proofs of the main results in the present paper proceeds by relating the position of the random walk to some expected hitting times. The latter are analyzed (over all environments) using estimates for sums of independent random variables; this relies on (mostly well-known) strong limit theorems.

We now give a formal description of the RWRE model that we study here. Fix $\delta \in (0, 1/2)$. Let (ξ_i, Y_i), $i \in \mathbb{N}$, be a sequence of i.i.d. random vectors on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that

$$\mathbb{P}[\delta \leq \xi_1 \leq 1 - \delta] = 1, \tag{2}$$

and Y_1 takes values in $[-1, 1]$. The condition (2) is sometimes referred to as *uniform ellipticity.* Note that we allow Y_1 and ξ_1 to be dependent.

We fix $\alpha > 0$. For a particular realization of the sequence (ξ_i, Y_i), $i \in \mathbb{N}$, we define $p_0 = q_0 = 1/2$ and the quantities p_n and q_n, $n = 1, 2, 3, \ldots$ as follows:

$$\begin{aligned} p_n &:= \begin{cases} \xi_n + Y_n n^{-\alpha} & \text{if} \quad (\delta/2) \leq \xi_n + Y_n n^{-\alpha} \leq 1 - (\delta/2) \\ \delta/2 & \text{if} \quad \xi_n + Y_n n^{-\alpha} < (\delta/2) \\ 1 - (\delta/2) & \text{if} \quad \xi_n + Y_n n^{-\alpha} > 1 - (\delta/2) \end{cases} \\ q_n &:= 1 - p_n. \end{aligned} \tag{3}$$

A particular realization of $(p_n; n \in \mathbb{N})$ specifies our random environment ω, and is given in terms of the ξ_i and Y_i as in (3). For a given environment ω, the stochastic process $(\eta_t(\omega); t \in \mathbb{Z}^+)$ as defined at (1) is an irreducible, aperiodic Markov chain (under P); the probability measure P in (1) is known as the *quenched* measure (the measure given a fixed environment ω).

Under condition (2), we have that there exists $n_0 \in \mathbb{N}$ such that, for a.e. ω, $(\delta/2) < \xi_n + Y_n n^{-\alpha} < 1 - (\delta/2)$ for all $n \geq n_0$ (since the Y_n are bounded). Thus, for all $n \geq n_0$, (3) implies that, for a.e. ω,

$$p_n = \xi_n + Y_n n^{-\alpha}, \quad q_n = 1 - \xi_n - Y_n n^{-\alpha}, \quad (n \geq n_0).$$

The conditions on the variables in (3) ensure that, for a.e. ω, $(\delta/2) \leq p_n \leq 1 - (\delta/2)$ for all n so that p_n and q_n are true probabilities bounded strictly away from 0 and 1, as required by our condition on ω given just before (1).

2 Main results

In this section we describe in detail two particular cases of the model formulated in the previous section, along with our main results in each case. Then in Section 2.3 we make further remarks and state some open problems.

2.1 Perturbation of random walk in random environment in Sinai's regime

Now we describe our first particular case of the model given in Section 1. For $n \in \mathbb{N}$ set

$$\zeta_n := \log \left(\frac{\xi_n}{1 - \xi_n} \right), \quad Z_n := \frac{Y_n}{\xi_n (1 - \xi_n)}. \tag{4}$$

With $\mathbb{E}$ denoting expectation under $\mathbb{P}$, suppose that $\mathbb{E}[\zeta_1] = 0$ and $\mathrm{Var}[\zeta_1] > 0$ (so our environment is truly random). In order to formulate our results, we introduce some more notation. Set

$$\lambda := \mathbb{E}[Z_1], \tag{5}$$

and also let

$$s^2 := \mathrm{Var}[\xi_1], \quad \sigma^2 := \mathrm{Var}[Y_1]. \tag{6}$$

Under our boundedness conditions on ξ_1 and Y_1, we have $s^2 < \infty$ and $\sigma^2 < \infty$, and under condition (2) we have, $\mathbb{P}$-a.s.,

$$-\infty < \frac{-1}{\delta^2} \leq Z_1 \leq \frac{1}{\delta^2} < \infty.$$

This model was introduced in [21] in somewhat more generality, and criteria for transience, recurrence and ergodicity given (see Theorems 6, 7 of [21]). In this case, the random environment described in (3) corresponds to a perturbation of Sinai's regime, in the sense that, in the limit as $n \to \infty$, we have $\mathbb{E}[\log(p_n/q_n)] \to 0$. Despite this, the behaviour of this model may be strikingly different to that of Sinai's RWRE (as demonstrated by our results below and also those in [21]), and depends on the sign of λ as defined at (5) (the average direction of the perturbation), and α (the size of the perturbation).

For the following results, with the definitions at (4) and (6), we take $s^2 > 0$, $\mathbb{E}[\zeta_1] = 0$, and $\sigma^2 \geq 0$ (so, for example, we permit the case $\mathbb{P}[Y_1 = b] = 1$ for some $b \in [-1, 1]$, i.e. a non-random perturbation of Sinai's RWRE). Of separate interest are the cases $\lambda = 0$ and $\lambda \neq 0$ (where λ is given by (5)). The case of most interest to us here is $\lambda \neq 0$, for which the perturbation is on average either towards 0 ($\lambda > 0$) or away from 0 ($\lambda < 0$); this includes the case of a non-random perturbation of Sinai's RWRE. It was shown in [21] that the critical size of the perturbation is $\alpha = 1/2$: for $\alpha < 1/2$ the perturbation is large enough to disturb the null-recurrent behaviour; for $\alpha \geq 1/2$ it is too small. By Theorem 6 of [21], we have that if $\lambda < 0$ and $\alpha < 1/2$ then $\eta_t(\omega)$ is transient for a.e. ω; if $\alpha \geq 1/2$ and $\lambda \neq 0$ then $\eta_t(\omega)$ is null-recurrent for a.e. ω; if $\lambda > 0$ and $\alpha < 1/2$ then $\eta_t(\omega)$ is ergodic for a.e. ω.

We obtain logarithmic speeds for the $\lambda \neq 0$ case, for the null-recurrent (Theorem 2), transient (Theorem 3), and ergodic (Theorem 5) regimes. In the case $\lambda = 0$, the critical exponent for α of $1/2$ is *decreased*, depending on certain higher order analogues of λ (see the remark after Theorem 7 of [21]). Here, of the $\lambda = 0$ cases, we will only be concerned (see Theorem 4, below) with the special case where $Y_1/\xi_1 \stackrel{\mathrm{d}}{=} -Y_1/(1-\xi_1)$, for which $\lambda = 0$ and $\eta_t(\omega)$ is null-recurrent for a.e. ω for *any* $\alpha > 0$ (see [21], Theorem 5). (Here and subsequently $\stackrel{\mathrm{d}}{=}$ stands for equality in distribution.) This case is of interest because, despite the presence of a (potentially strong) perturbation, the random walk remains null-recurrent; we show it has logarithmic speed.

Our first result is Theorem 2 below, which deals with the $\lambda \neq 0$, $\alpha \geq 1/2$ case, for which $\eta_t(\omega)$ is null-recurrent for a.e. ω (see above). Recall the definitions of λ, s^2 and σ^2 from (5) and (6).

Theorem 2 *Suppose* $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0, \infty)$, $\lambda \neq 0$ *and* $\sigma^2 \in [0, \infty)$.

(i) Suppose $\alpha > 1/2$. *Then, for a.e.* ω, *for any* $\varepsilon > 0$ *we have, a.s.,*

$$0 \leq \frac{\eta_t(\omega)}{(\log t)^2} < (\log\log t)^{2+\varepsilon}, \tag{7}$$

for all but finitely many t.

(ii) Suppose $\alpha = 1/2$. Then, for a.e. ω, for any $\varepsilon > 0$ we have, a.s.,

$$0 \le \frac{\eta_t(\omega)}{(\log t)^2} < (\log\log t)^{4+\varepsilon}, \tag{8}$$

for all but finitely many t.

(iii) On the other hand, for $\alpha \ge 1/2$, for a.e. ω, for any $\varepsilon > 0$ we have, a.s.,

$$\frac{\eta_t(\omega)}{(\log t)^2} > (\log\log\log t)^{-1-\varepsilon}, \tag{9}$$

for infinitely many t.

Our next result deals with the transient case when $\lambda < 0$ and $\alpha \in (0, 1/2)$, and gives a reasonably tight envelope which the random walk leaves only finitely often. Although the random walk is transient, it is very slow: we have a striking example of logarithmic transience.

Theorem 3 *Suppose $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0,\infty)$, $\lambda < 0$, $\sigma^2 \in [0,\infty)$, and $\alpha \in (0, 1/2)$. For a.e. ω, for any $\varepsilon > 0$, we have, a.s.,*

$$(\log\log t)^{-(1/\alpha)-\varepsilon} < \frac{\eta_t(\omega)}{(\log t)^{1/\alpha}} < (\log\log t)^{(2/\alpha)+\varepsilon}, \tag{10}$$

for all but finitely many t.

A case of secondary interest is that in which $Y_1/\xi_1 \stackrel{\mathrm{d}}{=} -Y_1/(1-\xi_1)$. Here $\lambda = 0$, and further, $\eta_t(\omega)$ is null-recurrent for a.e. ω, for any $\alpha > 0$ (see Theorem 5 of [21]). Our next result, Theorem 4 below, deals with this case.

The condition $Y_1/\xi_1 \stackrel{\mathrm{d}}{=} -Y_1/(1-\xi_1)$ ensures that although the perturbation may be strong, (roughly speaking) it balances out overall with equal strength to the left and to the right. This intuition is supported by the fact that the random walk remains null-recurrent. Also, included is the case $\mathbb{P}[Y_1 = 0] = 1$ and $\sigma^2 = 0$, i.e. Sinai's regime. Thus, for our purposes, there is no distinction between the behaviour of the RWRE perturbed from Sinai's regime under condition $Y_1/\xi_1 \stackrel{\mathrm{d}}{=} -Y_1/(1-\xi_1)$ and that of the RWRE in Sinai's regime itself.

Theorem 4 *Suppose $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0,\infty)$, $Y_1/\xi_1 \stackrel{\mathrm{d}}{=} -Y_1/(1-\xi_1)$, $\sigma^2 \in [0,\infty)$, and $\alpha > 0$.*

(i) For a.e. ω, for any $\varepsilon > 0$ we have that, a.s.,

$$0 \le \frac{\eta_t(\omega)}{(\log t)^2} \le (\log\log t)^{2+\varepsilon},$$

for all but finitely many t.

(ii) On the other hand, for a.e. ω, for any $\varepsilon > 0$ we have that, a.s.,

$$\frac{\eta_t(\omega)}{(\log t)^2} \ge (\log\log\log t)^{-1-\varepsilon},$$

for infinitely many t.

Remarks. (a) In the case of Sinai's regime ($\mathbb{P}[Y_1 = 0] = 1$, $\sigma^2 = 0$), Theorem 4(i) gives similar bounds to [4, 1], but by comparison to Theorem 1 (due to Hu and Shi [14]), none of the bounds in Theorem 4 is particularly sharp.
(b) In the null-recurrent regimes $\lambda \neq 0$, $\alpha \geq 1/2$ (Theorem 2) and $Y_1/\xi_1 \stackrel{\mathrm{d}}{=} -Y_1/(1-\xi_1)$ (Theorem 4) we see that the position of the random walk is essentially of order $(\log t)^2$, as in Sinai's regime (which is included in Theorem 4). Thus provided we have null-recurrence we have the same speed. On the other hand, in the transient case $\lambda < 0$, $\alpha < 1/2$ (Theorem 3), the $1/\alpha$ exponent in the speed of transience is in $(2, \infty)$. Thus for α increasingly small (i.e. a stronger perturbation), the speed increases (but is still 'slow', i.e. logarithmic).

In the ergodic situations, in addition to our results on the speed of the random walk, in the present paper we also give results on the rate of decay of the stationary distribution (π_n), $n \in \mathbb{Z}^+$, of the Markov chain $\eta_t(\omega)$. Some analogous results for non-random environments are given in [20]. Theorems 5 and 6 below deal with the ergodic case when $\lambda > 0$ and $\alpha \in (0, 1/2)$.

Theorem 5 *Suppose* $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0,\infty)$, $\lambda > 0$, $\sigma^2 \in [0,\infty)$, *and* $\alpha \in (0, 1/2)$. *For a.e.* ω, *for any* $\varepsilon > 0$, *a.s.,*

$$\eta_t(\omega) \leq (1+\varepsilon) \left(\frac{1-\alpha}{\lambda}\right)^{1/(1-\alpha)} (\log t)^{1/(1-\alpha)},$$

for all but finitely many t, *and*

$$\eta_t(\omega) \geq (1-\varepsilon) \left(\frac{1-\alpha}{\lambda}\right)^{1/(1-\alpha)} (\log t)^{1/(1-\alpha)},$$

for infinitely many t.

For $\alpha \in (0, 1/2)$, $1/(1-\alpha) \in (1, 2)$: this is 'slower' than Sinai's regime.

Theorem 6 *Suppose* $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0,\infty)$, $\lambda > 0$, $\sigma^2 \in [0,\infty)$, *and* $\alpha \in (0, 1/2)$. *For a.e.* ω, *as* $n \to \infty$

$$\pi_n = \exp\left(- \left(\frac{\lambda}{1-\alpha}\right) n^{1-\alpha} [1 + o(1)] \right). \tag{11}$$

2.2 Simple random walk with random perturbation

Our second model again fits into the framework of (3) above, but we now take $\mathbb{P}[\xi_1 = 1/2] = 1$ and $\sigma^2 := \mathrm{Var}[Y_1] > 0$. That is, we have a random perturbation of the symmetric simple random walk (SRW). In this case, from (3), we have $p_0 = q_0 = 1/2$ and for $n \in \mathbb{N}$

$$\begin{aligned} p_n &:= \begin{cases} \frac{1}{2} + Y_n n^{-\alpha} & \text{if} \quad (\delta/2) \leq \frac{1}{2} + Y_n n^{-\alpha} \leq 1 - (\delta/2) \\ \delta/2 & \text{if} \quad \frac{1}{2} + Y_n n^{-\alpha} < (\delta/2) \\ 1 - (\delta/2) & \text{if} \quad \frac{1}{2} + Y_n n^{-\alpha} > 1 - (\delta/2) \end{cases} \\ q_n &:= 1 - p_n. \end{aligned} \tag{12}$$

Since the Y_n are bounded, we have that there exists $n_0 \in \mathbb{N}$ such that for a.e. ω we have $(\delta/2) < \frac{1}{2} + Y_n n^{-\alpha} < 1 - (\delta/2)$ for all $n \geq n_0$. Thus, for a.e. ω, (12) implies that for all $n \geq n_0$

$$p_n = \frac{1}{2} + Y_n n^{-\alpha}, \quad q_n = \frac{1}{2} - Y_n n^{-\alpha}, \quad (n \geq n_0). \tag{13}$$

The conditions on the variables in (12) ensure that, for a.e. ω, $(\delta/2) \leq p_n \leq 1 - (\delta/2)$ for all n so that for p_n and q_n are bounded strictly away from 0 and 1. We see that, for a.e. ω, $(p_n, q_n) \to (1/2, 1/2)$ as $n \to \infty$. Thus in the limit $n \to \infty$, we coincide with the symmetric SRW on $\mathbb{Z}^+$.

Here we do not study the case $\mathrm{Var}[Y_1] = \sigma^2 = 0$, in which we have a non-random perturbation of the SRW. This is an example of the so-called Lamperti problem after [18] (see also [11]); for recurrence/transience criteria see [18, 19] and Theorem 2 of [21]. From now on we assume $\mathrm{Var}[Y_1] = \sigma^2 > 0$.

The transience and recurrence properties of the model given by (12) were analysed in [21]. From Theorem 3(iv) of [21], we have that in this case if $\mathbb{E}[Y_1] < 0$ and $\alpha < 1$ then $\eta_t(\omega)$ is transient for a.e. ω; if $\alpha > 1$ and $\mathbb{E}[Y_1] \neq 0$ then $\eta_t(\omega)$ is null-recurrent for a.e. ω; if $\mathbb{E}[Y_1] > 0$ and $\alpha < 1$ then $\eta_t(\omega)$ is ergodic for a.e. ω. Thus, in contrast to the perturbation of the *random* environment (as in Section 2.1), the critical exponent in this case is $\alpha = 1$.

When $\mathbb{E}[Y_1] = 0$, recurrence/transience properties depend on the higher moments of Y_1 (see the remark after Theorem 3 of [21]). Of interest to us in the present paper is the case in which the distribution of Y_1 is symmetric, that is $Y_1 \stackrel{\mathrm{d}}{=} -Y_1$ (and $\mathbb{E}[Y_1] = 0$). In this case (see Theorem 3(iii) of [21]) $\eta_t(\omega)$ is null-recurrent for a.e. ω, for *any* $\alpha > 0$. In this case we obtain our logarithmic behaviour (see Theorem 7), in the domain $\alpha \in (0, 1/2)$. We also obtain logarithmic bounds in the ergodic case mentioned above (see Theorem 8).

Theorem 7 *Suppose* $\mathbb{P}[\xi_1 = 1/2] = 1$, $Y_1 \stackrel{\mathrm{d}}{=} -Y_1$, $\sigma^2 \in (0, \infty)$, $\alpha \in (0, 1/2)$.

(i) For a.e. ω, for any $\varepsilon > 0$, a.s.,

$$0 \leq \frac{\eta_t(\omega)}{(\log t)^{2/(1-2\alpha)}} \leq (\log \log t)^{(2/(1-2\alpha))+\varepsilon}, \tag{14}$$

for all but finitely many t.

(ii) On the other hand, for a.e. ω, for any $\varepsilon > 0$, a.s.,

$$\frac{\eta_t(\omega)}{(\log t)^{2/(1-2\alpha)}} \geq (\log \log \log t)^{-(1/(1-2\alpha))-\varepsilon}, \tag{15}$$

for infinitely many t.

Remark. Note that for $\alpha \in (0, 1/2)$, $2/(1-2\alpha)$ is in $(2, \infty)$. In the limit $\alpha \downarrow 0$, we approach Sinai's regime in the sense that, for fixed ω and each n, $(p_n, q_n) \to (\frac{1}{2} + Y_n, \frac{1}{2} - Y_n)$ where

$$\mathbb{E}\left[\log\left(\frac{(1/2) + Y_n}{(1/2) - Y_n}\right)\right] = \mathbb{E}[\log((1/2) + Y_n)] - \mathbb{E}[\log((1/2) - Y_n)] = 0$$

when $Y_1 \stackrel{d}{=} -Y_1$. Thus it is not surprising that in the limit $\alpha \downarrow 0$, Theorem 7 approaches Theorem 4 (which includes Sinai's regime).

Theorems 8 and 9 below deal with the ergodic case when $\mathbb{E}[Y_1] > 0$ and $\alpha \in (0,1)$. Note that when $\alpha \in (0,1)$, $1/(1-\alpha) \in (1,\infty)$.

Theorem 8 *Suppose* $\mathbb{P}[\xi_1 = 1/2] = 1$, $\mathbb{E}[Y_1] > 0$, $\sigma^2 \in (0,\infty)$, *and* $\alpha \in (0,1)$. *For a.e.* ω, *for any* $\varepsilon > 0$, *a.s.,*

$$\eta_t(\omega) \leq (1+\varepsilon)\left(\frac{1-\alpha}{4\mathbb{E}[Y_1]}\right)^{1/(1-\alpha)} (\log t)^{1/(1-\alpha)},$$

for all but finitely many t, *and*

$$\eta_t(\omega) \geq (1-\varepsilon)\left(\frac{1-\alpha}{4\mathbb{E}[Y_1]}\right)^{1/(1-\alpha)} (\log t)^{1/(1-\alpha)},$$

for infinitely many t.

The next result gives the rate of decay of the stationary distribution (π_n): as in Theorem 6, the decay is sub-exponential.

Theorem 9 *Suppose* $\mathbb{P}[\xi_1 = 1/2] = 1$, $\mathbb{E}[Y_1] > 0$, $\sigma^2 \in (0,\infty)$, *and* $\alpha \in (0,1)$. *For a.e.* ω, *as* $n \to \infty$

$$\pi_n = \exp\left(-\left(\frac{4\mathbb{E}[Y_1]}{1-\alpha}\right) n^{1-\alpha}[1+o(1)]\right). \tag{16}$$

2.3 Further remarks and open problems

Our results give an indication of the 'almost sure' behaviour of $\eta_t(\omega)$, and there is scope for tightening our bounds. Also of interest is the so-called *annealed* behaviour of the RWRE (averaged over all environments). Sinai's result [27] for the random walk in i.i.d. random environment on $\mathbb{Z}$ with $\mathbb{E}[\log(p_1/q_1)] = 0$ showed (roughly speaking) that $\eta_t(\omega)$ divided by $(\log t)^2$ converges in distribution to some random variable as $t \to \infty$. The result is stated in terms of the annealed probability measure $\mathbb{Q}$ given by

$$\mathbb{Q}[\cdot] = \int_\Omega P[\cdot] \mathrm{d}\mathbb{P}[\omega].$$

Golosov [8] showed that for the RWRE on $\mathbb{Z}^+$ in Sinai's regime

$$\mathbb{Q}\left[\frac{\eta_t(\omega)}{(\log t)^2} \leq u\right] \longrightarrow F(u), \quad u \in \mathbb{R},$$

as $t \to \infty$, where F is a known distribution function. See also [9, 10, 15, 2] for related results. The annealed behaviour of our models is also of interest. In particular, under the conditions of Theorem 3 do we have (analogously to the results of Sinai-Golosov [27, 8]) that as $t \to \infty$

$$\mathbb{Q}\left[\frac{\eta_t(\omega)}{(\log t)^{1/\alpha}} \leq u\right] \longrightarrow G(u), \quad u \in \mathbb{R},$$

for some G? We do not address this question in the present paper.

One can obtain L^p analogues of our results, with the methods used here (compare Theorem 3.2 of [1]). For example, under the conditions of Theorem 2, analogously to (7), for any $p \geq 1$, for any $\varepsilon > 0$, for a.e. ω, as $t \to \infty$

$$\frac{\eta_t(\omega)}{(\log t)^{2+\varepsilon}} \to 0, \text{ in } L^p.$$

The methods of the present paper are well suited to logarithmic speeds, since they are based on an analysis of the expected hitting times of the random walk; some standard estimates using the submartingale property, Markov's inequality and the (first) Borel-Cantelli lemma lead to some rather sharp results, since these expected times are exponentially large. Of interest would be results for the cases of the SRW with random perturbation that are not covered by the theorems of Section 2.2. For example, if $Y_1 \stackrel{\mathrm{d}}{=} -Y_1$ but $\alpha > 1/2$, we expect SRW-like behaviour. On the other hand, if $\mathbb{E}[Y_1] \neq 0$, we suspect that $\eta_t(\omega)$ will behave in a similar way to the Lamperti problem mentioned above: roughly speaking, we expect SRW-like behaviour for $\alpha > 1$, while in the transient regime ($\alpha < 1$ and $\mathbb{E}[Y_1] < 0$) we have $\eta_t(\omega) \sim t^{1/(1+\alpha)}$. Another open problem is the behaviour of this model when $\alpha = 1$ (this case was not covered in [21]). We hope to address some of these issues in future work.

3 Preliminaries

Before we prove our main results in Section 4, we give some preparatory results. First, in Section 3.1, we present some technical lemmas concerning the behaviour of sums of independent random variables; some are well-known results, others we prove. Then, in Section 3.2, we give the main apparatus of our proofs, based on some hitting time results.

3.1 Some strong theorems for sums of independent random variables

The following result is due to Sakhanenko [23, 24, 25], and is contained in Theorem A* of the more readily obtainable paper by Shao [26].

Lemma 1 *Let $X_1, X_2, \ldots$ be independent random variables with $E[X_i] = 0$, $\mathrm{Var}[X_i] = \sigma_i^2 \in (0, \infty)$ for $i \in \mathbb{N}$. Suppose that the X_i are uniformly bounded, i.e., for some $B \in (0, \infty)$, $P[|X_i| > B] = 0$ for all i. For $n \in \mathbb{N}$, set*

$$s_n^2 := \sum_{i=1}^{n} \sigma_i^2.$$

Then, there exists (possibly on an enlarged probability space) a sequence of independent normal random variables $(W_1, \ldots, W_n)$ with $E[W_i] = 0$, $\mathrm{Var}[W_i] = \sigma_i^2$ for $1 \leq i \leq n$ such that a.s.,

$$\left| \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} W_i \right| \leq \frac{1}{A} \log(s_n^2),$$

for all but finitely many n, where $A \in (0, \infty)$ is a constant.

We will need a form of the Law of the Iterated Logarithm. The following result is a consequence of Theorem 7 of [7].

Lemma 2 *Let $X_1, X_2, \ldots$ be independent, uniformly bounded random variables with $E[X_i] = 0$, $\mathrm{Var}[X_i^2] = \sigma_i^2 \in (0, \infty)$ for $i \in \mathbb{N}$. For $n \in \mathbb{N}$, set $s_n^2 := \sum_{i=1}^n \sigma_i^2$. Suppose that $s_n \to \infty$ as $n \to \infty$. Then, for any $\varepsilon > 0$, a.s.,*

$$\left| \sum_{i=1}^{n} X_i \right| \leq s_n((2+\varepsilon) \log\log(s_n^2))^{1/2},$$

for all but finitely many n.

We will also need the following 'inverse iterated logarithm law' due to Hirsch (Theorem 2 of [12]; see also Theorem 3.1 of [3]).

Lemma 3 *Let $X_1, X_2, \ldots$ be i.i.d., uniformly bounded random variables with $E[X_1] = 0$, $\mathrm{Var}[X_1] \in (0, \infty)$. For $x \geq 0$, let $a(x) > 0$ be a nonincreasing function such that $x^{1/2}a(x)$ is eventually increasing and*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n} < \infty. \tag{17}$$

Then, a.s.,

$$\max_{1 \leq i \leq n} \sum_{j=1}^{i} X_j \geq n^{1/2} a(n),$$

for all but finitely many n.

We will also need the following extension of part of Hirsch's result to independent non-identically distributed random variables.

Lemma 4 *Let $X_1, X_2, \ldots$ be independent, uniformly bounded random variables with $E[X_i] = 0$, $\mathrm{Var}[X_i] = \sigma_i^2$ for $i \in \mathbb{N}$, where $0 < \sigma_i^2 < M < \infty$ for all i. Set $s_n^2 := \sum_{i=1}^n \sigma_i^2$ for $n \in \mathbb{N}$. Suppose that $s_n \to \infty$ as $n \to \infty$. For $x \geq 0$, let $a(x) > 0$ be a nonincreasing function such that $x^{1/2}a(x)$ is eventually increasing, (17) holds, and*

$$\lim_{n \to \infty} \frac{\log n}{n^{1/2} a(n)} = 0. \tag{18}$$

Then, for some constant $C \in (0, \infty)$, a.s.,

$$\max_{1 \leq i \leq n} \sum_{j=1}^{i} X_j \geq C s_n a(s_n^2), \tag{19}$$

for all but finitely many n.

Proof. By Lemma 1, we can redefine the X_i, $i \in \mathbb{N}$ on a richer probability space along with a sequence of independent normal random variables W_i, $i \in \mathbb{N}$ with $E[W_i] = 0$ and $\mathrm{Var}[W_i] = \sigma_i^2$, such that, a.s.,

$$\left| \sum_{j=1}^{i} X_j - \sum_{j=1}^{i} W_j \right| \leq A^{-1} \log(s_i^2) \leq C \log i,$$

for all but finitely many i, for some $A, C \in (0, \infty)$. Thus, a.s.,

$$\left| \max_{1 \leq i \leq n} \sum_{j=1}^{i} X_j - \max_{1 \leq i \leq n} \sum_{j=1}^{i} W_j \right| \leq \max_{1 \leq i \leq n} \left| \sum_{j=1}^{i} X_j - \sum_{j=1}^{i} W_j \right| \leq C \log n, \tag{20}$$

for all but finitely many n. For $n \in \mathbb{N}$, set

$$h(n) := \min\{m \in \mathbb{N} : s_m^2 \geq n\}. \tag{21}$$

There exists a standard Brownian motion $(B(n); n \geq 0)$ and a sequence of independent normal random variables $\delta_n \sim \mathcal{N}(0, s_{h(n)}^2 - n)$, $n \in \mathbb{N}$, independent of $(B(n); n \geq 0)$, such that

$$B(n) + \delta_n = \sum_{i=1}^{h(n)} W_i,$$

for each $n \in \mathbb{N}$. Now,

$$\max_{1 \leq i \leq h(n)} \sum_{j=1}^{i} W_j \geq \max_{1 \leq i \leq n} \sum_{j=1}^{h(i)} W_j = \max_{1 \leq i \leq n} \left(B(i) + \delta_i \right).$$

Hence

$$\max_{1 \leq i \leq h(n)} \sum_{j=1}^{i} W_j \geq \max_{1 \leq i \leq n} B(i) - \max_{1 \leq i \leq n} \delta_i. \tag{22}$$

Since $\mathrm{Var}(\delta_i) = s_{h(i)}^2 - i \leq \sigma_{h(i)}^2 < M < \infty$, and δ_i, $i \in \{1, \ldots, n\}$ are independent normal random variables, we have that, a.s.,

$$\max_{1 \leq i \leq n} \delta_i \leq \log n, \tag{23}$$

for all but finitely many n (this follows from standard tail bounds on the normal distribution (see e.g. [5], p. 9) and the Borel-Cantelli lemma). Suppose that $a(\cdot)$ satisfies the conditions of this lemma. Now, for a sequence $Y_1, Y_2, \ldots$ of i.i.d. normal random variables with $E[Y_1] = 0$ and $\mathrm{Var}[Y_1] = 1$, we have by Lemma 3 that a.s.,

$$\max_{1 \leq i \leq n} B(i) = \max_{1 \leq i \leq n} \sum_{j=1}^{i} Y_j \geq n^{1/2} a(n), \tag{24}$$

for all but finitely many n. So from (22), (23), (24) and condition (18), a.s.,

$$\max_{1 \leq i \leq h(n)} \sum_{j=1}^{i} W_j \geq n^{1/2} a(n) - \log n \geq C n^{1/2} a(n), \tag{25}$$

for all but finitely many n and some $C \in (0, \infty)$. Since $\sigma_i^2 > 0$ for all i, we have from (21) that $h(s_n^2) = n$; thus by (20) and (25) we have that, a.s.,

$$\max_{1 \le i \le n} \sum_{j=1}^{i} X_j \ge \max_{1 \le i \le n} \sum_{j=1}^{i} W_j - C \log n \ge C'(s_n^2)^{1/2} a(s_n^2) - C \log n,$$

for some $C, C' \in (0, \infty)$, for all but finitely many n. Then by the conditions on s_n^2 and $a(\cdot)$, (19) follows. □

The next two lemmas will be needed for some more delicate estimates (e.g. in the proof of Theorem 3) where we need to deal with certain moving sums. The following lemma is a corollary to a result of Hirsch [12].

Lemma 5 *Let $X_1, X_2, \ldots$ be independent, uniformly bounded random variables with $E[X_i] = 0$, $\mathrm{Var}[X_i] = \sigma^2 \in (0, \infty)$ for $i \in \mathbb{N}$. For $x \ge 1$, let $b(x)$ be a nondecreasing, integer-valued function such that for some $\beta > 0$ and $x_0 \in (0, \infty)$, $x^\beta \le b(x) \le x$ for all $x \ge x_0$. Then for any $\varepsilon > 0$, a.s.,*

$$\max_{1 \le i \le n} \max_{1 \le j \le b(i)} \sum_{k=i-j+1}^{i} X_k \ge (b(n/2))^{1/2} (\log n)^{-1-\varepsilon},$$

for all but finitely many n.

Proof. For fixed i, note that

$$\max_{1 \le j \le b(i)} \sum_{k=i-j+1}^{i} X_k \stackrel{\mathrm{d}}{=} \max_{1 \le j \le b(i)} \sum_{k=1}^{j} Y_k,$$

where $Y_1, Y_2, \ldots$ are independent random variables with $Y_k \stackrel{\mathrm{d}}{=} X_{i+1-k}$ for each k. Fix $\varepsilon > 0$. Let E_i denote the event

$$E_i := \left\{ \max_{1 \le j \le b(i)} \sum_{k=i-j+1}^{i} X_k \le (b(i))^{1/2} (\log b(i))^{-1-\varepsilon} \right\}.$$

Then Corollary 1 of Hirsch [12] implies that there are absolute constants $C, C' \in (0, \infty)$ such that for all $i \ge x_0$,

$$P[E_i] \le C(\log b(i))^{-1-\varepsilon} \le C'(\log i)^{-1-\varepsilon},$$

since $b(i) \ge i^\beta$. Consider the subsequence $i = 2^m$ for $m = 1, 2, \ldots$. Then

$$\sum_{m=1}^{\infty} P[E_{2^m}] \le C \sum_{m=1}^{\infty} m^{-1-\varepsilon} < \infty.$$

Hence by the (first) Borel-Cantelli lemma, a.s., there is a finite m_0 (with $2^{m_0} \ge x_0$) such that, for all $i = 2^m$ with $m \ge m_0$,

$$\max_{1 \le j \le b(i)} \sum_{k=i-j+1}^{i} X_k \ge (b(i))^{1/2} (\log b(i))^{-1-\varepsilon} \ge (b(i))^{1/2} (\log i)^{-1-\varepsilon},$$

since $b(i) \leq i$. Each $n \geq 2$ satisfies $n \in [2^m, 2^{m+1})$ for some $m \in \mathbb{N}$; then, a.s.,

$$\max_{1 \leq i \leq n} \max_{1 \leq j \leq b(i)} \sum_{k=i-j+1}^{i} X_k \geq \max_{1 \leq i \leq 2^m} \max_{1 \leq j \leq b(i)} \sum_{k=i-j+1}^{i} X_k$$
$$\geq \max_{1 \leq j \leq b(2^m)} \sum_{k=2^m-j+1}^{2^m} X_k \geq (b(2^m))^{1/2} (\log(2^m))^{-1-\varepsilon},$$

provided $m \geq m_0$. Hence, since $n \geq 2^m > n/2$, a.s.,

$$\max_{1 \leq i \leq n} \max_{1 \leq j \leq b(i)} \sum_{k=i-j+1}^{i} X_k \geq (b(n/2))^{1/2} (\log n)^{-1-\varepsilon},$$

for all $n \geq 2^{m_0}$. □

Lemma 6 *Let $X_1, X_2, \ldots$ be independent, uniformly bounded random variables with $E[X_i] = 0$ for all $i \in \mathbb{N}$. Then there exists $C \in (0, \infty)$ such that, a.s., for all but finitely many i*

$$\left| \sum_{k=i-j+1}^{i} X_k \right| \leq C j^{1/2} (\log i)^{1/2},$$

for all $j = 1, 2, \ldots, i$.

Proof. For fixed i, $Y^i_j := \sum_{k=i-j+1}^{i} X_k$ is a martingale over $j = 1, 2, \ldots, i$, with uniformly bounded increments. Hence the Azuma-Hoeffding inequality (see e.g. [13]) implies that for some $B \in (0, \infty)$, for all $j = 1, \ldots, i$, for $t > 0$,

$$P[|Y^i_j| \geq t] \leq 2 \exp(-B^{-1} j^{-1} t^2).$$

Thus for a suitable $C < \infty$, for $j \leq i$, $P[|Y^i_j| \geq C j^{1/2} (\log i)^{1/2}] \leq i^{-3}$. Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i} P[|Y^i_j| \geq C j^{1/2} (\log i)^{1/2}] \leq \sum_{i=1}^{\infty} i^{-2} < \infty.$$

Hence the (first) Borel-Cantelli lemma implies that, a.s., there are only finitely many pairs (i, j) (with $j \leq i$) for which $|Y^i_j| \geq C j^{1/2} (\log i)^{1/2}$. □

3.2 Hitting times results

For the proofs of our main results, we will use the expected hitting times for the random walk $\eta_t(\omega)$ as defined at (1). For the remainder of this section, we work in the quenched setting (i.e. with fixed environment $\omega = (p_0, p_1, \ldots)$ throughout). For $0 \leq m < n$, let $\tau_{m,n}$ denote the time when $\eta_t(\omega)$ first hits n, starting from m. That is, with the convention $\min \emptyset = +\infty$,

$$\tau_{m,n} := \min\{t \geq 0 : \eta_t(\omega) = n | \eta_0(\omega) = m\}. \tag{26}$$

For our proofs in Section 4, we take $\eta_0(\omega) = r = 0$ for ease of exposition; the proofs easily extend to general $r \in \mathbb{Z}^+$. For fixed ω, let $T(0) := 0$, and for $n \in \mathbb{N}$ let $T(n) := E[\tau_{0,n}]$. For $i = 0, 1, 2, \ldots$, write $\Delta_i := T(i+1) - T(i) = E[\tau_{i,i+1}]$, so that Δ_i is the expected time taken for $\eta_t(\omega)$ to hit $i+1$, starting at i. Then standard arguments yield $T(n) = \sum_{i=0}^{n-1} \Delta_i$ with $\Delta_0 = 1/q_0$ and for $i \geq 1$

$$\Delta_i = 1 + p_i(\Delta_{i-1} + \Delta_i).$$

We then obtain the following classical result.

Lemma 7 *Let ω be fixed. For $n \in \mathbb{N}$, we have that $T(n) = \sum_{i=0}^{n-1} \Delta_i$, and for $i \geq 0$, Δ_i is given (with the convention that an empty product is 1) by*

$$\Delta_i \;=\; \sum_{j=0}^{i} q_{i-j}^{-1} \prod_{k=i-j+1}^{i} \frac{p_k}{q_k} = \frac{1}{q_i} + \frac{p_i}{q_i q_{i-1}} + \cdots + \frac{p_i p_{i-1} \cdots p_1}{q_i q_{i-1} \cdots q_1 q_0}. \tag{27}$$

The following fact will be very useful. That is, for a fixed environment, $T(\eta_t(\omega))$ is a submartingale with respect to the natural filtration (for a closely related supermartingale, see [1], equation (6)). In particular, we have the following.

Lemma 8 *For fixed ω, any $t \in \mathbb{Z}^+$ and any $n \in \mathbb{Z}^+$,*

$$E[T(\eta_{t+1}(\omega)) - T(\eta_t(\omega)) | \eta_t(\omega) = n] = 1. \tag{28}$$

Proof. For $n \geq 1$, we have

$$\begin{aligned}
& E[T(\eta_{t+1}(\omega)) - T(\eta_t(\omega)) | \eta_t(\omega) = n] \\
= \;& p_n(T(n-1) - T(n)) + q_n(T(n+1) - T(n)) \\
= \;& q_n \Delta_n - p_n \Delta_{n-1} = 1,
\end{aligned}$$

by (27). Also,

$$E[T(\eta_{t+1}(\omega)) - T(\eta_t(\omega)) | \eta_t(\omega) = 0] = q_0 T(1) = 1,$$

since $T(1) = \Delta_0 = 1/q_0$. $\square$

We can now state the result that will be our main tool in proving almost sure upper and lower bounds for $\eta_t(\omega)$, using the expected hitting times $T(n)$.

Lemma 9 *For a given environment ω, suppose that there exist two nonnegative, increasing, continuous functions g and h such that,*

$$g(n) \leq T(n) \leq h(n),$$

for all $n \in \mathbb{Z}^+$. Then:

(i) For any $\varepsilon > 0$, a.s., for all but finitely many t,

$$\eta_t(\omega) \leq g^{-1}((2t)^{1+\varepsilon}). \tag{29}$$

(ii) A.s., for infinitely many t,

$$(\eta_t(\omega))^2 h(\eta_t(\omega)) \geq t. \tag{30}$$

Remark. In the transient case we want to do better (for Theorem 3) than part (ii) here, to give a lower bound for $\eta_t(\omega)$ that holds all but finitely often. See the proof of Theorem 3 below.

Proof of Lemma 9. Throughout we work in fixed environment ω. First we prove part (i). From (28), we have that for any $t \in \mathbb{Z}^+$

$$E[T(\eta_{t+1}(\omega)) - T(\eta_t(\omega))] = \sum_{n=0}^{\infty} P[\eta_t(\omega) = n] = 1.$$

Then, given that $\eta_0(\omega) = 0$, for all $t \in \mathbb{Z}^+$ we have

$$E[T(\eta_t(\omega))] = t. \tag{31}$$

To prove (29), we modify the idea of the proof of Theorem 3.2 of [1]. Since $T(\eta_t(\omega))$ is a nonnegative submartingale (see Lemma 8), Doob's submartingale inequality (see e.g. [29], p. 137) implies that, for $t > 0$, for any $\varepsilon > 0$,

$$P\left[\max_{0 \le s \le t} T(\eta_s(\omega)) \ge t^{1+\varepsilon}\right] \le t^{-1-\varepsilon} E[T(\eta_t(\omega))] = t^{-\varepsilon}, \tag{32}$$

using (31). Also, given that $T(n) \ge g(n)$ for all n, we have, for $t > 0$,

$$\begin{aligned} P\left[\max_{0 \le s \le t} T(\eta_s(\omega)) \ge t^{1+\varepsilon}\right] &\ge P\left[\max_{0 \le s \le t} g(\eta_s(\omega)) \ge t^{1+\varepsilon}\right] \\ &= P\left[g\left(\max_{0 \le s \le t} \eta_s(\omega)\right) \ge t^{1+\varepsilon}\right], \end{aligned} \tag{33}$$

since g is increasing. Hence from (32) and (33), for $t > 0$,

$$P\left[\max_{0 \le s \le t} \eta_s(\omega) \ge g^{-1}(t^{1+\varepsilon})\right] \le t^{-\varepsilon}.$$

Thus along the subsequence $t = 2^m$ for $m = 0, 1, 2, \ldots$, the (first) Borel-Cantelli lemma implies that, a.s., the event in the last display occurs only finitely often, and in particular there exists $m_0 < \infty$ such that for all $m \ge m_0$

$$\max_{0 \le s \le 2^m} \eta_s(\omega) \le g^{-1}((2^m)^{1+\varepsilon}).$$

Every t sufficiently large has $2^m \le t < 2^{m+1}$ for some $m \ge m_0$; then, a.s.,

$$\eta_t(\omega) \le \max_{0 \le s \le t} \eta_s(\omega) \le \max_{0 \le s \le 2^{m+1}} \eta_s(\omega) \le g^{-1}((2^{m+1})^{1+\varepsilon}),$$

for all but finitely many t. Now since $2^{m+1} \le 2t$ and g^{-1} is increasing, (29) follows.

Now we prove part (ii). Recall the definition of $\tau_{0,n}$ at (26). By Markov's inequality, we have that for $n \in \mathbb{N}$

$$P[\tau_{0,n} > n^2 T(n)] = P[\tau_{0,n} > n^2 E[\tau_{0,n}]] \le n^{-2}.$$

Then, by the (first) Borel-Cantelli lemma, a.s., $\tau_{0,n} > n^2 T(n)$ for only finitely many n. Thus, given that $T(n) \le h(n)$ for all n, we have that a.s., for all but finitely many n, $\tau_{0,n} \le n^2 h(n)$.

Given ω, $\eta_t(\omega)$ is an irreducible Markov chain on $\mathbb{Z}^+$, hence $\limsup_{t\to\infty} \eta_t(\omega) = +\infty$ a.s.. Thus a.s. there exists an infinite subsequence of $\mathbb{N}$, $t_1, t_2, t_3, \ldots$ (one can take, for each i, $t_i = \tau_{0,i}$, the time of the first visit of η_t to i), such that $\eta_{t_i}(\omega) \to \infty$ as $i \to \infty$. That is, a.s.,

$$t_i \leq \eta_{t_i}(\omega)^2 h(\eta_{t_i}(\omega)).$$

There are infinitely many such t_i, and so we have (30). □

4 Proofs of main results

To prove our main results, we employ the machinery given in the previous section: we obtain, via the results in Section 3.1, suitable functions g, h such that $g(n) \leq T(n) \leq h(n)$ (for a.e. ω), and then apply Lemma 9.

We consider $T(n)$ as given in Lemma 7. Recalling the definition of Δ_i at (27), we can write (interpreting an empty sum as zero) for $i \geq 0$

$$\Delta_i = \sum_{j=0}^{i} q_{i-j}^{-1} \exp \sum_{k=i-j+1}^{i} \log(p_k/q_k). \tag{34}$$

The following result gives general bounds on $T(n)$.

Lemma 10 *For a fixed environment ω, for all $n \geq 1$*

$$T(n) \geq \exp \max_{1\leq i\leq n-1} \sum_{k=1}^{i} \log(p_k/q_k), \tag{35}$$

and for some $C \in (0, \infty)$, for all $n \geq 1$,

$$T(n) \leq Cn^2 \exp\left(\max_{0\leq i\leq n-1} \sum_{k=1}^{i} \log(p_k/q_k) + \max_{0\leq i\leq n-1} \sum_{k=1}^{i} (-\log(p_k/q_k)) \right). \tag{36}$$

Proof. Since a sum of nonnegative terms is bounded below by its largest term,

$$T(n) = \sum_{i=0}^{n-1} \Delta_i \geq \max_{1\leq i\leq n-1} \Delta_i \geq \max_{1\leq i\leq n-1} \max_{1\leq j\leq i} \exp \sum_{k=i-j+1}^{i} \log(p_k/q_k), \tag{37}$$

using (34) and the fact that $q_{i-j}^{-1} \geq 1$. Now for $i \in \mathbb{N}$

$$\max_{1\leq j\leq i} \sum_{k=i-j+1}^{i} \log(p_k/q_k) \geq \sum_{k=1}^{i} \log(p_k/q_k), \tag{38}$$

so that by (37) and (38),

$$T(n) \geq \max_{1\leq i\leq n-1} \exp \max_{1\leq j\leq i} \sum_{k=i-j+1}^{i} \log(p_k/q_k) \geq \max_{1\leq i\leq n-1} \exp \sum_{k=1}^{i} \log(p_k/q_k),$$

and the lower bound in the lemma follows.

For the upper bound, we have from (34) that

$$T(n) \leq n \max_{0\leq i\leq n-1} \Delta_i \leq \delta^{-1} n(n+1) \max_{0\leq i\leq n-1} \max_{0\leq j\leq i} \exp \sum_{k=i-j+1}^{i} \log(p_k/q_k), \tag{39}$$

since $q_{i-j}^{-1} \leq \delta^{-1}$ with δ as at (2). Now

$$\begin{aligned}
\max_{0\leq j\leq i} \sum_{k=i-j+1}^{i} \log(p_k/q_k) &= \sum_{k=1}^{i} \log(p_k/q_k) + \max_{0\leq j\leq i} \sum_{k=1}^{i-j} (-\log(p_k/q_k)) \\
&= \sum_{k=1}^{i} \log(p_k/q_k) + \max_{0\leq j\leq i} \sum_{k=1}^{j} (-\log(p_k/q_k)).
\end{aligned} \tag{40}$$

Thus from (39) and (40), for $C \in (0, \infty)$ and all $n \geq 1$

$$T(n) \leq Cn^2 \exp\left(\max_{0\leq i\leq n-1} \sum_{k=1}^{i} \log(p_k/q_k) + \max_{0\leq i\leq n-1} \max_{0\leq j\leq i} \sum_{k=1}^{j} (-\log(p_k/q_k)) \right).$$

Then the upper bound in the lemma follows. □

We start with the proof of Theorem 7 for expository purposes. The proof of Theorem 7 will then serve as a prototype for subsequent proofs. As previously mentioned, we take $\eta_0(\omega) = 0$ for the purposes of the proofs that follow (without loss of generality).

4.1 Proof of Theorem 7

For fixed ω, by Lemma 7, the expected hitting time $T(n)$ is expressed in terms of $\log(p_n/q_n)$. To prepare for the proof, we note that under the conditions of Theorem 7 p_n and q_n have the same distribution, so

$$\mathbb{E}[\log(p_n/q_n)] = \mathbb{E}[\log p_n] - \mathbb{E}[\log q_n] = 0. \tag{41}$$

By (13), Taylor's theorem and the boundedness of the Y_n, for a.e. ω,

$$\begin{aligned}
\log p_n &= \log(1/2) + \log(1 + 2Y_n n^{-\alpha}) \\
&= \log(1/2) + 2Y_n n^{-\alpha} - 2Y_n^2 n^{-2\alpha} + O(n^{-3\alpha}),
\end{aligned}$$

for all n sufficiently large, and

$$\begin{aligned}
\log q_n &= \log(1/2) + \log(1 - 2Y_n n^{-\alpha}) \\
&= \log(1/2) - 2Y_n n^{-\alpha} - 2Y_n^2 n^{-2\alpha} + O(n^{-3\alpha}),
\end{aligned}$$

so that

$$\log(p_n/q_n) = \log p_n - \log q_n = 4Y_n n^{-\alpha} + O(n^{-3\alpha}). \tag{42}$$

Lemma 11 below gives bounds for the expected hitting time $T(n)$, and so prepares us for the proof of Theorem 7 via an application of Lemma 9.

Lemma 11 *Suppose* $\mathbb{P}[\xi_1 = 1/2] = 1$, $Y_1 \stackrel{\mathrm{d}}{=} -Y_1$, $\sigma^2 \in (0,\infty)$, *and* $\alpha \in (0,1/2)$. *Then for a.e.* ω, *for any* $\varepsilon > 0$, *for all but finitely many* n,

$$\exp(n^{(1-2\alpha)/2}(\log n)^{-1}(\log\log n)^{-1-\varepsilon}) \leq T(n) \\ \leq \exp(n^{(1-2\alpha)/2}(\log\log n)^{(1/2)+\varepsilon}). \tag{43}$$

Proof. From (41), $\mathbb{E}[\log(p_k/q_k)] = 0$ and from (42) $\mathrm{Var}[\log(p_k/q_k)] = 16\sigma^2 k^{-2\alpha} + o(k^{-2\alpha})$. Hence, for $\alpha \in (0,1/2)$, for all i,

$$C_1 i^{1-2\alpha} \leq \mathrm{Var} \sum_{k=1}^{i} \log(p_k/q_k) \leq C_2 i^{1-2\alpha}, \tag{44}$$

for some $C_1, C_2 \in (0,\infty)$ with $C_1 < C_2$.

Now we derive the lower bound in (43). By Lemma 4 and (44), for an appropriate choice of $a(\cdot)$ satisfying the conditions of Lemma 4, for a.e. ω, a.s.,

$$\max_{1\leq i\leq n-1} \sum_{k=1}^{i} \log(p_k/q_k) \geq C n^{(1-2\alpha)/2} a(n^{1-2\alpha}), \tag{45}$$

for all but finitely many n. For $\varepsilon > 0$, we take $a(n) = (\log n)^{-1}(\log\log n)^{-1-\varepsilon}$; then $a(\cdot)$ satisfies the conditions of Lemma 4. Then (35) and (45) imply the lower bound in (43).

Now we prove the upper bound in (43), using (36). By Lemma 2 with (44) we have that for a.e. ω, a.s., for all but finitely many n,

$$\max_{0\leq i\leq n-1} \sum_{k=1}^{i} \log(p_k/q_k) < C n^{(1-2\alpha)/2}(\log\log n)^{1/2},$$

$$\max_{0\leq i\leq n-1} \sum_{k=1}^{i} (-\log(p_k/q_k)) < C n^{(1-2\alpha)/2}(\log\log n)^{1/2},$$

for some $C \in (0,\infty)$. Thus from (36) we obtain the upper bound in (43). □

Proof of Theorem. First we prove part (i) of Theorem 7. From the lower bound in (43), we have that, for a.e. ω, there exists a finite positive constant C (depending on ω) such that, for any $\varepsilon > 0$, for all n sufficiently large,

$$T(n) \geq g(n) := C\exp\left(n^{(1-2\alpha)/2}(\log n)^{-1}(\log\log n)^{-1-\varepsilon}\right). \tag{46}$$

So by (29), we have that, for a.e. ω, for any $\varepsilon > 0$, a.s.,

$$\eta_t(\omega) \leq g^{-1}(4t^2) \leq C((\log t)(\log\log t)^{1+\varepsilon})^{2/(1-2\alpha)},$$

for all but finitely many t, which gives (14). Now we prove part (ii). From the upper bound in (43), we have that, for any $\varepsilon > 0$,

$$T(n) \leq h(n) := C\exp(n^{(1-2\alpha)/2}(\log\log n)^{(1/2)+\varepsilon}),$$

so that, for all n sufficiently large,

$$h^{-1}(n) \geq C(\log n)^{2/(1-2\alpha)}(\log\log\log n)^{-(1+3\varepsilon)/(1-2\alpha)}. \tag{47}$$

From (30) we have that a.s., for infinitely many t,

$$h(\eta_t(\omega)) \geq t(\eta_t(\omega))^{-2} \geq Ct(\log t)^{-5/(1-2\alpha)},$$

by (14). Thus a.s., for infinitely many t, $\eta_t(\omega) \geq h^{-1}(Ct(\log t)^{-5/(1-2\alpha)})$, which with (47) yields (15). □

4.2 Proofs of Theorems 2 and 3

To prove Theorems 2 and 3, we proceed along the same lines as the proof of Theorem 7 in Section 4.1, and apply Lemma 9. Theorem 3 (the transient case) requires some extra work, both to obtain suitable bounds for $T(n)$ and to prove that the lower bound on the random walk holds all but finitely often.

Suppose $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0,\infty)$, $\sigma^2 \in [0,\infty)$. Then for a.e. ω

$$\log\left(\frac{p_n}{q_n}\right) = \zeta_n + \log\left(1 + \frac{Y_n}{\xi_n} n^{-\alpha}\right) - \log\left(1 - \frac{Y_n}{1-\xi_n} n^{-\alpha}\right), \tag{48}$$

for all $n \geq n_0$ for a finite absolute constant n_0, where ζ_i, $i \in \mathbb{N}$, as defined at (4) are i.i.d. with $\mathbb{E}[\zeta_1] = 0$ and $\mathrm{Var}[\zeta_1] \in (0,\infty)$. It follows from (48) and Taylor's theorem that, for a.e. ω, for all n sufficiently large,

$$\log(p_n/q_n) = \zeta_n + Z_n n^{-\alpha} + O(n^{-2\alpha}), \tag{49}$$

where Z_i, $i \in \mathbb{N}$, are i.i.d. with $\mathbb{E}[Z_1] = \lambda$ (see (4) and (5)). Then by (49)

$$\mathbb{E}[\log(p_k/q_k)] = \lambda k^{-\alpha} + O(k^{-2\alpha}), \mathrm{Var}[\log(p_k/q_k)] = \mathrm{Var}[\zeta_1] + O(k^{-\alpha}). \tag{50}$$

Lemma 12 *Suppose $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0,\infty)$, $\sigma^2 \in [0,\infty)$, $\lambda < 0$, and $\alpha \in (0,1/2)$. For a.e. ω and any $\varepsilon > 0$, for all but finitely many n,*

$$\exp(n^\alpha (\log n)^{-2-\varepsilon}) \leq T(n) \leq \exp(n^\alpha (\log n)^{1+\varepsilon}). \tag{51}$$

Proof. First we prove the upper bound in (51). Since $\lambda < 0$, we have from (50) that

$$\mathbb{E} \sum_{k=i-j+1}^{i} \log(p_k/q_k) \leq -C(i^{1-\alpha} - (i-j)^{1-\alpha}), \tag{52}$$

for some $C \in (0,\infty)$. Taylor's theorem implies that for $\alpha \in (0,1)$

$$i^{1-\alpha} - (i-j)^{1-\alpha} = Cji^{-\alpha}(1-\theta(j/i))^{-\alpha}, \tag{53}$$

for some $C \in (0,\infty)$ and $\theta \in (0,1)$. Thus it follows from (52) and (53) that for all $i \in \mathbb{N}$, and all $j = 1, 2, \ldots, i$

$$\mathbb{E} \sum_{k=i-j+1}^{i} \log(p_k/q_k) \leq -Cji^{-\alpha}, \tag{54}$$

for some $C \in (0,\infty)$. By Lemma 6 we have that, for some $C \in (0,\infty)$, for a.e. ω, all but finitely many i, and all $j = 1, 2, \ldots, i$,

$$\sum_{k=i-j+1}^{i} (\log(p_k/q_k) - \mathbb{E}[\log(p_k/q_k)]) \leq Cj^{1/2}(\log i)^{1/2}. \tag{55}$$

Suppose $\varepsilon > 0$. Then from (55) with (54), for a.e. ω, for $j \geq \lceil i^{2\alpha}(\log i)^{1+\varepsilon} \rceil$

$$\sum_{k=i-j+1}^{i} \log(p_k/q_k) \leq -Cji^{-\alpha} + C'j^{1/2}(\log i)^{1/2} \leq -C''ji^{-\alpha}, \tag{56}$$

and, for $j \leq \lceil i^{2\alpha}(\log i)^{1+\varepsilon} \rceil$

$$\sum_{k=i-j+1}^{i} \log(p_k/q_k) \leq C j^{1/2} (\log i)^{1/2}, \tag{57}$$

where each inequality holds for all but finitely many i. So from (34), (56) and (57) we obtain, for a.e. ω, for any $\varepsilon > 0$, for all but finitely many i,

$$\begin{aligned} \Delta_i &\leq \sum_{j=0}^{\lceil i^{2\alpha}(\log i)^{1+\varepsilon} \rceil} \exp(C j^{1/2} (\log i)^{1/2}) + \sum_{j=\lceil i^{2\alpha}(\log i)^{1+\varepsilon} \rceil}^{i} \exp(-C' j i^{-\alpha}) \\ &\leq \exp(C'' i^{\alpha} (\log i)^{1+\varepsilon}), \end{aligned}$$

for $C'' \in (0, \infty)$. Then the upper bound for $T(n)$ in (51) follows.

We now prove the lower bound in (51). For $\varepsilon > 0$ set $k_\varepsilon(1) := 1$ and for $i > 1$ define

$$k_\varepsilon(i) := \lfloor i^{2\alpha} (\log i)^{-2-\varepsilon} \rfloor. \tag{58}$$

Then, for any $\alpha \in (0, 1/2]$ and all n sufficiently large, from (37),

$$T(n) \geq \max_{1 \leq i \leq n-1} \max_{1 \leq j \leq k_\varepsilon(i)} \exp \sum_{k=i-j+1}^{i} \log(p_k/q_k). \tag{59}$$

Then (48) and Taylor's theorem imply that there is a constant $C \in (0, \infty)$ such that, for all k, $\log(p_k/q_k) = \zeta_k + W_k k^{-\alpha}$, where $|W_k| < C$. Thus for $i \in \mathbb{N}$ and $j = 1, 2, \ldots, i$,

$$\sum_{k=i-j+1}^{i} \log(p_k/q_k) \geq \sum_{k=i-j+1}^{i} \zeta_k - C \sum_{k=i-j+1}^{i} k^{-\alpha} \geq \sum_{k=i-j+1}^{i} \zeta_k - C' j i^{-\alpha},$$

again using Taylor's theorem (cf (53)). Hence by (59)

$$T(n) \geq \exp\left(\max_{1 \leq i \leq n-1} \max_{1 \leq j \leq k_\varepsilon(i)} \sum_{k=i-j+1}^{i} \zeta_k - C k_\varepsilon(n) n^{-\alpha} \right). \tag{60}$$

By Lemma 5, we have that for any $\varepsilon > 0$, for a.e. ω,

$$\max_{1 \leq i \leq n-1} \max_{1 \leq j \leq k_\varepsilon(i)} \sum_{k=i-j+1}^{i} \zeta_k \geq (k_\varepsilon(n/2))^{1/2} (\log n)^{-1-(\varepsilon/4)} \geq C n^{\alpha} (\log n)^{-2-(3\varepsilon/4)},$$

for all but finitely many n, while $k_\varepsilon(n) n^{-\alpha} \leq n^{\alpha} (\log n)^{-2-\varepsilon}$. Hence (60) implies the lower bound in (51). $\square$

Lemma 13 *Suppose* $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0, \infty)$, $\sigma^2 \in [0, \infty)$, *and* $\lambda \neq 0$.

(i) Suppose $\alpha > 1/2$. *For a.e.* ω *and any* $\varepsilon > 0$, *for all but finitely many* n,

$$\exp(n^{1/2} (\log n)^{-1-\varepsilon}) \leq T(n) \leq \exp(n^{1/2} (\log \log n)^{(1/2)+\varepsilon}). \tag{61}$$

(ii) Suppose $\alpha = 1/2$. For a.e. ω and any $\varepsilon > 0$, for all but finitely many n,

$$\exp(n^{1/2}(\log n)^{-2-\varepsilon}) \leq T(n) \leq \exp(n^{1/2}(\log \log n)^{(1/2)+\varepsilon}). \tag{62}$$

Proof. To prove the upper bounds in (61) and (62), we apply (36). For $\lambda \neq 0$, $\alpha \geq 1/2$ we have from (50) that

$$\sum_{k=1}^{i} \mathbb{E}[\log(p_k/q_k)] = O(\max\{i^{1-\alpha}, \log i\}),$$

so that for some $C \in (0, \infty)$ and all n

$$\max_{0 \leq i \leq n} \sum_{k=1}^{i} \log(p_k/q_k) \leq \max_{0 \leq i \leq n} \sum_{k=1}^{i} (\log(p_k/q_k) - \mathbb{E}[\log(p_k/q_k)]) + C \max\{n^{1-\alpha}, \log n\}; \tag{63}$$

similarly for the second maximum in (36). By Lemma 2 and (50), for a.e. ω,

$$\max_{0 \leq i \leq n} \sum_{k=1}^{i} (\log(p_k/q_k) - \mathbb{E}[\log(p_k/q_k)]) \leq C n^{1/2} (\log \log n)^{1/2},$$

for all but finitely many n, and since $\alpha \geq 1/2$, (63) then implies that for a.e. ω,

$$\max_{0 \leq i \leq n} \sum_{k=1}^{i} \log(p_k/q_k) \leq C n^{1/2} (\log \log n)^{1/2},$$

for all but finitely many n, and similarly for the second maximum in (36). Then (36) gives the upper bounds in (61) and (62).

Now we prove the lower bounds in (61) and (62). In the case $\alpha > 1/2$,

$$\max_{1 \leq i \leq n-1} \sum_{k=1}^{i} \log(p_k/q_k) \geq \max_{1 \leq i \leq n-1} \sum_{k=1}^{i} (\log(p_k/q_k) - \mathbb{E}[\log(p_k/q_k)]) - C \max\{n^{1-\alpha}, \log n\},$$

by a similar argument to (63). Lemma 4 implies that for any $\varepsilon > 0$, for a.e. ω,

$$\max_{1 \leq i \leq n-1} \sum_{k=1}^{i} (\log(p_k/q_k) - \mathbb{E}[\log(p_k/q_k)]) \geq n^{1/2} (\log n)^{-1-\varepsilon},$$

for all but finitely many n; then (35) implies the lower bound in (61). Finally, suppose $\alpha = 1/2$. Once more we define $k_\varepsilon(i)$ by (58), and follow the argument for (60). This yields the lower bound in (62). □

Proof of Theorem 3. For the upper bound in (10), the lower bound in (51) implies that, for a.e. ω, for any $\varepsilon > 0$ there exists $C \in (0, \infty)$ such that

$$T(n) \geq g(n) := C \exp(n^\alpha (\log n)^{-2-\varepsilon}),$$

for all n sufficiently large. Then (29) gives, for a.e. ω, for any $\varepsilon > 0$, a.s.,

$$\eta_t(\omega) \leq g^{-1}(4t^2) \leq C (\log t)^{1/\alpha} (\log \log t)^{(2+2\varepsilon)/\alpha},$$

for all but finitely many t. Then the upper bound in (10) follows.

We now want to obtain the lower bound in (10). Recalling the proof of Lemma 9(ii), we were able to show that, along a sequence of first hitting times for the random walk, these times were not too large. This gave us a lower bound that was valid infinitely often. In order to extend this technique to the transient case, and obtain a lower bound valid *all but finitely* often, we show in addition that (roughly speaking), in the present case, the time of the last visit of the random walk to a site is not too much greater than the first hitting time.

For fixed ω, let a_n denote the probability that the random walk $\eta_t(\omega)$ hits n in finite time, given that it starts at $2n$. For $n \geq 1$ define

$$M_n := 1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \frac{p_{n+k}}{q_{n+k}} = 1 + \sum_{j=1}^{\infty} \exp \sum_{k=1}^{j} \log \left(\frac{p_{n+k}}{q_{n+k}} \right). \tag{64}$$

Standard hitting probability arguments yield $a_0 = 1$, and for $n \geq 1$, if $M_n < \infty$,

$$a_n = M_n^{-1} \sum_{j=n}^{\infty} \prod_{k=1}^{j} \frac{p_{n+k}}{q_{n+k}} = M_n^{-1} \sum_{j=n}^{\infty} \exp \sum_{k=1}^{j} \log \left(\frac{p_{n+k}}{q_{n+k}} \right). \tag{65}$$

In the present case ($\lambda < 0$, $\alpha \in (0, 1/2)$), (50) holds. Thus for $n, j \in \mathbb{N}$

$$\mathbb{E} \sum_{k=1}^{j} \log(p_{n+k}/q_{n+k}) \leq -C((n+j)^{1-\alpha} - n^{1-\alpha}), \tag{66}$$

for some $C \in (0, \infty)$. Here, by Taylor's theorem, for $\alpha \in (0, 1)$,

$$(n+j)^{1-\alpha} - n^{1-\alpha} = Cj(n + \theta j)^{-\alpha}, \tag{67}$$

for some $C \in (0, \infty)$, $\theta \in (0, 1)$. In particular, for $j \geq n$, (66) and (67) imply

$$\mathbb{E} \sum_{k=1}^{j} \log(p_{n+k}/q_{n+k}) \leq -Cj(j(1+\theta))^{-\alpha} \leq -C' j^{1-\alpha}, \tag{68}$$

for $C' \in (0, \infty)$. Also, by the Azuma-Hoeffding inequality and an argument similar to Lemma 6, we have that, for a.e. ω,

$$\sum_{k=1}^{j} (\log(p_{n+k}/q_{n+k}) - \mathbb{E}[\log(p_{n+k}/q_{n+k})]) \leq Cj^{1/2}(\log(jn))^{1/2},$$

for all but finitely many (n, j). Thus for all (n, j) we have that, for a.e. ω,

$$\sum_{k=1}^{j} \log(p_{n+k}/q_{n+k}) \leq Cj^{1/2}(\log(jn))^{1/2}, \tag{69}$$

for some $C \in (0, \infty)$. However, we have from (68) that, for a.e. ω, there are constants $C, C', C'' \in (0, \infty)$ such that, for all $n \in \mathbb{N}$, and $j \geq n$

$$\sum_{k=1}^{j} \log(p_{n+k}/q_{n+k}) \leq -Cj^{1-\alpha} + C' j^{1/2} (\log j)^{1/2} \leq -C'' j^{1-\alpha}, \tag{70}$$

since $\alpha \in (0, 1/2)$. Hence, for a.e. ω, from (64), (69), and (70), for $n \in \mathbb{N}$,

$$M_n \leq \sum_{j=1}^{n} \exp(Cj^{1/2}(\log(jn))^{1/2}) + \sum_{j=n}^{\infty} \exp(-C'j^{1-\alpha})$$
$$\leq \exp(C''n^{1/2}(\log n)^{1/2}) < \infty.$$

Further, since $M_n \geq 1$ for all n, (65) and (70) imply, for a.e. ω, for all $n \in \mathbb{N}$,

$$a_n \leq \sum_{j=n}^{\infty} \exp(-Cj^{1-\alpha}) \leq \exp(-C'n^{1-\alpha}),$$

for some $C' \in (0, \infty)$. Thus, for a.e. ω, $\sum_n a_n < \infty$.

The (first) Borel-Cantelli lemma then implies that, for a.e. ω, a.s., for only finitely many sites n does $\eta_t(\omega)$ return to n after visiting $2n$. Denoting by ℓ_n the time of the last visit of $\eta_t(\omega)$ to n, we then have that $\ell_n \leq \tau_{0,2n}$ a.s. for all but finitely many n. Suppose $T(n) \leq h(n)$ for all n. Following the proof of Lemma 9(ii), we have that a.s., for all but finitely many n, $\tau_{0,n} \leq n^2 h(n)$. Thus for a.e. ω, a.s.,

$$\ell_n \leq \tau_{0,2n} \leq 4n^2 h(2n), \tag{71}$$

for all $n \geq n_0$ for some finite n_0 (depending on ω).

Moreover, since, for a.e. ω, $\eta_t(\omega)$ is transient, we have that, for a.e. ω, a.s., $\eta_t(\omega) \geq n_0$ for all t sufficiently large. Hence from (71), using the fact that $\ell_{\eta_t(\omega)} \geq t$ for all t, we have that for a.e. ω, a.s., for all but finitely many t,

$$t \leq \ell_{\eta_t(\omega)} \leq 4\eta_t(\omega)^2 h(2\eta_t(\omega)).$$

Then, with the upper bound in (51), we obtain, for a.e. ω, for any $\varepsilon > 0$, a.s.,

$$t < \exp(\eta_t(\omega)^\alpha (\log \eta_t(\omega))^{1+\varepsilon}),$$

for all but finitely many t. This implies the lower bound in (10). $\square$

Proof of Theorem 2. We first prove part (i). Suppose $\alpha > 1/2$. From the lower bound on $T(n)$ in (61), for a.e. ω, for any $\varepsilon > 0$,

$$T(n) \geq g(n) := C \exp(n^{1/2}(\log n)^{-1-\varepsilon}),$$

for all $n \in \mathbb{N}$. Then (29) implies the upper bound in (7). For part (ii), when $\alpha = 1/2$, the lower bound in (62) allows us, this time, to take $g(n) := C \exp(n^{1/2}(\log n)^{-2-\varepsilon})$. Then (29) gives the upper bound in (8).

For part (iii) of the theorem, for $\alpha \geq 1/2$, the upper bound on $T(n)$ in (61) and (62) implies that for a.e. ω

$$T(n) \leq h(n) := C \exp(n^{1/2}(\log \log n)^{(1/2)+\varepsilon}),$$

for all but finitely many n; in particular h^{-1} satisfies the lower bound of (47) with $\alpha = 0$. Then (30) yields the lower bound in (9). $\square$

4.3 Proofs of Theorems 5 and 8

We now move on to the ergodic cases (Theorems 5 and 8). Again we start by bounding $T(n)$. First we deal with the ergodic case of the random perturbation of the simple random walk.

Lemma 14 *Suppose* $\mathbb{P}[\xi_1 = 1/2] = 1$, $\mathbb{E}[Y_1] > 0$, $\sigma^2 \in (0, \infty)$, *and* $\alpha \in (0, 1)$. *Then for a.e.* ω, *as* $n \to \infty$

$$T(n) = \exp\left(\frac{4\mathbb{E}[Y_1]}{1-\alpha} n^{1-\alpha}[1 + o(1)]\right).$$

Proof. In this case, (42) holds. We apply a variation of the argument for Lemma 6. We have that for each i

$$Y_j^i := \sum_{k=i-j+1}^{i} (\log(p_k/q_k) - \mathbb{E}[\log(p_k/q_k)])$$

is a martingale over $j = 1, 2, \ldots, i$, with increments $|Y_j^i - Y_{j-1}^i|$ bounded by

$$|\log(p_{i-j+1}/q_{i-j+1})| + |\mathbb{E}[\log(p_{i-j+1}/q_{i-j+1})]| \leq C(i - j + 1)^{-\alpha} =: c_j^i,$$

for some $C \in (0, \infty)$, by (42). Thus for each $j \leq i$, for $\alpha \in (0, 1)$,

$$\sum_{k=1}^{j} (c_k^i)^2 = C \sum_{k=1}^{j} (i - k + 1)^{-2\alpha} \leq C' i^{1-\alpha}.$$

Then for each i and $j \leq i$ the Azuma-Hoeffding inequality implies that

$$\mathbb{P}[|Y_j^i| \geq t] \leq 2\exp(-Ct^2 i^{\alpha-1}),$$

for all $t > 0$. Hence for any $\varepsilon > 0$, the Borel-Cantelli lemma implies that

$$\max_{1 \leq j \leq i} |Y_j^i| \leq i^{((1-\alpha)/2)+\varepsilon},$$

for all but finitely many i. Also, from (42),

$$\mathbb{E} \sum_{k=i-j+1}^{i} \log(p_k/q_k) = \frac{4\mathbb{E}[Y_1]}{1-\alpha} \left(i^{1-\alpha} - (i-j)^{1-\alpha}\right) [1 + o(1)].$$

Hence for all i sufficiently large, since $\varepsilon > 0$ was arbitrary and $\alpha \in (0, 1)$

$$\sum_{k=i-j+1}^{i} \log(p_k/q_k) = \frac{4\mathbb{E}[Y_1]}{1-\alpha} \left(i^{1-\alpha} - (i-j)^{1-\alpha}\right) [1 + o(1)] + o(i^{1-\alpha}). \tag{72}$$

Thus from (34) and (72), as $i \to \infty$,

$$\begin{aligned} \Delta_i &= \exp\left(\frac{4\mathbb{E}[Y_1]}{1-\alpha} i^{1-\alpha}[1 + o(1)]\right) \sum_{j=0}^{i} \exp\left(-Cj^{1-\alpha}[1 + o(1)]\right) \\ &= \exp\left(\frac{4\mathbb{E}[Y_1]}{1-\alpha} i^{1-\alpha}[1 + o(1)]\right), \end{aligned}$$

from which the lemma follows. □

Proof of Theorem 8. Once again we apply Lemma 9. First we prove the lower bound. From Lemma 14 we have that for a.e. ω, for all n

$$T(n) \geq g(n) := \exp\left(\frac{4\mathbb{E}[Y_1]}{1-\alpha} n^{1-\alpha}[1+o(1)]\right).$$

It follows that

$$g^{-1}(n) = \left(\frac{1-\alpha}{4\mathbb{E}[Y_1]}\right)^{1/(1-\alpha)} (\log n)^{1/(1-\alpha)}[1+o(1)].$$

Then (29) implies that a.s., for all but finitely many t, for any $\varepsilon > 0$

$$\begin{aligned} \eta_t(\omega) &\leq g^{-1}((2t)^{1+\varepsilon}) \\ &= (1+\varepsilon)^{1/(1-\alpha)} \left(\frac{1-\alpha}{4\mathbb{E}[Y_1]}\right)^{1/(1-\alpha)} (\log t)^{1/(1-\alpha)}[1+o(1)], \end{aligned}$$

and thus we obtain the upper bound in the theorem. On the other hand, Lemma 14 implies that for a.e. ω, any $\varepsilon > 0$, and all n

$$T(n) \leq h(n) := \exp\left(\frac{4\mathbb{E}[Y_1]}{1-\alpha} n^{1-\alpha}[1+o(1)]\right).$$

Then (30) implies that, a.s., for infinitely many t,

$$t \leq (\eta_t(\omega))^2 h(\eta_t(\omega)) \leq \exp\left(\frac{4\mathbb{E}[Y_1]}{1-\alpha}(\eta_t(\omega))^{1-\alpha}[1+o(1)]\right),$$

from which the lower bound in the theorem follows. □

Now we deal with the ergodic case of the perturbation of Sinai's regime.

Lemma 15 *Suppose $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0, \infty)$, $\lambda > 0$, $\sigma^2 \in [0, \infty)$, and $\alpha \in (0, 1/2)$. Then for a.e. ω, as $n \to \infty$*

$$T(n) = \exp\left(\frac{\lambda}{1-\alpha} n^{1-\alpha}[1+o(1)]\right).$$

Proof. In this case, we have that (50) holds (now with $\lambda > 0$). Thus

$$\mathbb{E} \sum_{k=i-j+1}^{i} \log(p_k/q_k) = \frac{\lambda}{1-\alpha}\left(i^{1-\alpha} - (i-j)^{1-\alpha}\right)[1+o(1)].$$

Now we can apply Lemma 6 to obtain for a.e. ω, for all but finitely many i,

$$\left| \sum_{k=i-j+1}^{i} (\log(p_k/q_k) - \mathbb{E}[\log(p_k/q_k)]) \right| \leq C j^{1/2} (\log i)^{1/2},$$

for $j = 1, 2, \ldots, i$. Since $\alpha < 1/2$ we have that for a.e. ω, as $i \to \infty$

$$\sum_{k=i-j+1}^{i} \log(p_k/q_k) = \frac{\lambda}{1-\alpha}\left(i^{1-\alpha} - (i-j)^{1-\alpha}\right)[1 + o(1)]. \tag{73}$$

Hence, from (34) and (73), as $i \to \infty$,

$$\begin{aligned} \Delta_i &= \exp\left(\frac{\lambda}{1-\alpha} i^{1-\alpha}[1 + o(1)]\right) \sum_{j=0}^{i} \exp\left(-Cj^{1-\alpha}[1 + o(1)]\right) \\ &= \exp\left(\frac{\lambda}{1-\alpha} i^{1-\alpha}[1 + o(1)]\right), \end{aligned}$$

and so the lemma follows. □

Proof of Theorem 5. The proof follows in a similar way to the above proof of Theorem 8, this time using the bounds in Lemma 15 and applying Lemma 9 once more. □

4.4 Proof of Theorem 4

We now prove Theorem 4. Once more, with the definition of ζ_n and Z_n at (4), we have that for a.e. ω and n sufficiently large, $\log(p_n/q_n)$ is given by (48), (49). In this case $\mathbb{E}[\zeta_1] = 0$ and $Y_1/\xi_1 \stackrel{\mathrm{d}}{=} -Y_1/(1-\xi_1)$, which implies that $\mathbb{E}[\log(p_n/q_n)] = 0$ for all n sufficiently large.

Lemma 16 *Suppose $\mathbb{E}[\zeta_1] = 0$, $s^2 \in (0, \infty)$, $Y_1/\xi_1 \stackrel{\mathrm{d}}{=} -Y_1/(1-\xi_1)$, $\sigma^2 \in [0, \infty)$, and $\alpha > 0$. For a.e. ω, for any $\varepsilon > 0$, for all but finitely many n,*

$$\exp(n^{1/2}(\log n)^{-1-\varepsilon}) \leq T(n) \leq \exp(n^{1/2}(\log \log n)^{(1/2)+\varepsilon}). \tag{74}$$

Proof. We apply Lemma 10. We have that (49) and (50) hold in this case. For the upper bound, consider (36). By Lemma 2 we have that for a.e. ω, for all but finitely many n,

$$\max_{0 \leq i \leq n-1} \sum_{k=1}^{i} \log(p_k/q_k) < Cn^{1/2}(\log \log n)^{1/2},$$

for some $C \in (0, \infty)$, and similarly for the second maximum in (36). Then (36) implies the upper bound in (74). For the lower bound, we use (35). We apply Lemma 4 with $a(x) = (\log x)^{-1-\varepsilon}$ to obtain, for a.e. ω, for any $\varepsilon > 0$

$$\max_{1 \leq i \leq n-1} \sum_{k=1}^{i} \log(p_j/q_j) \geq n^{1/2}(\log n)^{-1-\varepsilon},$$

for all but finitely many n. With (35), the lower bound in (74) follows. □

Proof of Theorem. Again the proof is very similar to that of Theorems 7 and 2, this time using Lemma 16 and Lemma 9. □

4.5 Proofs of Theorems 6 and 9

Finally, we prove the results on the stationary distribution in the ergodic cases given in Theorems 6 and 9. Given ω, suppose $\eta_t(\omega)$ is ergodic; then there exists a unique stationary distribution $(\pi_0, \pi_1, \pi_2, \ldots)$. It is straightforward to obtain the result (see, for example, Lemma 5 of [21]) that, for a given ω such that $\eta_t(\omega)$ is ergodic, there exists a constant $C \in (0, \infty)$ such that, for all $n \geq 2$,

$$\pi_n = C \prod_{k=1}^{n} \frac{q_k}{p_k} = C \exp\left(\sum_{k=1}^{n} \log(q_k/p_k) \right). \tag{75}$$

Proof of Theorem 6. Here we have that $\log(p_n/q_n)$ is given by (49), with $\lambda > 0$, $\alpha \in (0, 1/2)$ and $\mathbb{E}[\zeta_1] = 0$. In this case the $j = i = n$ case of (73) implies that

$$\sum_{k=1}^{n} \log(q_k/p_k) = -\sum_{k=1}^{n} \log(p_k/q_k) = -\frac{\lambda}{1-\alpha} n^{1-\alpha}[1 + o(1)],$$

as $n \to \infty$. Then (75) yields (11). □

Proof of Theorem 9. This time we have that $\log(p_n/q_n)$ is given by (42), where now $\mathbb{E}[Y_1] > 0$ and $\alpha \in (0, 1)$. In this case the $j = i = n$ case of (72) implies that

$$\sum_{k=1}^{n} \log(q_k/p_k) = -\sum_{k=1}^{n} \log(p_k/q_k) = -\frac{4\mathbb{E}[Y_1]}{1-\alpha} n^{1-\alpha}[1 + o(1)],$$

as $n \to \infty$. Then (75) yields (16). □

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